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International Journal of Solids and Structures 42 (2005) 1151–1171
INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES
www.elsevier.com/locate/ijsolstr

Thermo-electro-elastic transient responses in piezoelectric hollow structures

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Received 5 January 2004; received in revised form 17 June 2004
Available online 3 September 2004

Abstract

The paper presents an analytical method to solve thermo-electro-elastic transient response in piezoelectric hollow structures subjected to arbitrary thermal shock, sudden mechanical load and electric excitation. Volterra integral equation of the second kind caused by interaction between elastic deformation and electric field is solved by using an interpolation method. Thus, the exact expressions for the transient responses of displacement, stresses, electric displacement and electric potential in the piezoelectric hollow structures are obtained by means of Hankel transform, Laplace transform, and their inverse transforms. In Section 2, based on spherical coordinates, the governing equation of thermo-electro-elastic transient responses in a piezoelectric hollow sphere is found and the associated numerical results are carried out. In Section 3, based on cylindrical coordinates, the governing equation of thermo-electro-elastic transient responses in a non-homogeneous piezoelectric hollow cylinder is found and the corresponding numerical results are carried out. The results carried out may be used as a reference to solve other transient coupled problems of thermo-electro-elasticity in piezoelectric structures.

Keywords: Thermo-electro-elastic; Piezoelectric hollow sphere; Piezoelectric hollow cylinder; Non-homogeneous; Transient response

1. Introduction

In recent years, there is an accelerated effort and notable contributions on the study of thermo-electro-elastic coupling behavior in some engineering areas, including aerospace, offshore and submarine

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0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2004.06.061

Nomenclature

ε_{ij}	component of strains
u_r	radial displacement [m]
c_{ij}, e_{ij}, α_i, p_{ij}	elastic constants [N/m^2], piezoelectric constants [C/m^2], thermal expansion coefficients [1/k] and dielectric constants [C^2/N m^2]
λ_i, β_{ii}	thermal modulus [N/m^2 K], and pyroelectric coefficient [C/m^2 K]
σ_{ij}, D_{rr}	the component of stresses [N/m^2] and radial electric displacement [C/m^2]
$\psi(r,t)$	electric potential [W/A]
$T(r,t)$	temperature change function [k]
ρ, t	mass density [kg/m^3] and time variable [s]
r	radial variable [m]
a, b	inner and outer radii of piezoelectric hollow sphere or cylinder [m]
C_L	electro-elastic wave speed [m/s]
ω	inherent frequency of piezoelectric hollow structures [1/s]

structures, chemical vessel and civil engineering structures. These structures can be simplified to a transversely isotropic hollow sphere or an orthotropic hollow cylinder, and can be easily exposed to a variety of temperature fields in different environments. The understanding of mechanical behaviors of piezoelectric structures is thus of significant importance.

In previous investigations of piezoelectric structures, there are some investigations on hollow sphere. For piezoelectric materials, in Sinha (1962), the static solution of radial deformation of a piezoelectric spherical shell under uniform pressures on the internal and external surfaces, and subjected to a given voltage difference between these surface, coupled with a radial distribution of temperature was successfully solved. Dynamic thermal shock in a hollow sphere and spherical elastic shell of arbitrary thickness was investigated by Zaker (1966, 1968). Parida and Das (1972) studied the transient thermal stresses in a homogeneous orthotropic thin circular disc due to an instantaneous point heat source. Sugano (1979) solved the transient thermal stresses in a homogeneous transversely isotropic, finite cylinder due to an arbitrary internal heat generation. Due to a constant temperature imposed on one surface and heat convection into a medium at the other surface, Kardomateas (1989, 1990) obtained the transient thermal stresses in a homogeneous hollow cylinder. Shul'ga (1990) studied the radial electro-elastic vibrations of a hollow piezoceramic sphere. Thermal shock in a hollow sphere caused by rapid uniform heating was analyzed by Hata (1991). Wang (1995, 1996) studied the dynamic thermal stresses in homogeneous isotropic solid cylinders and hollow cylinder subjected to thermal shock. Abd-Alla (1995) solved the thermal stresses in a homogeneous, transversely isotropic, infinite cylindrical shell subjected to an instantaneous heat source. Wang et al. (2001) obtained dynamic thermal stress in a transversely isotropic hollow sphere. Chen and Shioya (2001) investigated the piezothermoelastic behavior of a pyroelectric spherical shell. Ding et al. (2003) investigated dynamic response of a pyroelectric hollow sphere under radial deformation.

The other hand, many studies on transient responses of non-homogeneous structures have been also done. Shaffer (1967) obtained the general solutions for a non-homogeneous orthotropic annular disk in plane stress subjected to uniform pressures at the internal and external surface. The torsional oscillations of a finite non-homogeneous piezoelectric cylindrical shell were investigated by Sarma (1980). Abd-Alla et al. (1999) studied the transient thermal stresses in a rotating non-homogeneous cylindrically orthotropic composite tube and in a non-homogeneous spherically orthotropic elastic medium with spherical cavity, respectively. Horgan and Chan (1999) investigated the pressured FGM hollow cylinder and disk problems. Tarn (2001) obtained an exact solution of functionally graded anisotropic cylinders subjected to thermal and mechanical loads for a steady-state problem. The electro-elastic problems for a special non-homogeneous piezoelectric hollow cylinder had been studied by Hou et al. (2003). The non-homogeneous material has gained much attention because of its good heat-shielding character as well as other significant superiorities. To date, investigations on the interactions of thermo-electro-mechanical coupled behavior in homogeneous piezoelectric structure have mainly considered static interactions between thermal, electric and mechanical fields and transient interaction between electric field and mechanical field in a non-homogeneous structure.

However, investigations on thermo-electro-elastic transient response of a transversely isotropic piezoelectric hollow sphere and thermo-electro-elastic transient response in a non-homogeneous orthotropic piezoelectric hollow cylinder, subjected to arbitrary thermal shock, sudden mechanical load and electric excitation have been few.

This paper presents an exact solution for thermo-electro-elastic transient response in piezoelectric hollow structures subjected to arbitrary thermal shock, radial shock load and electric excitation. The thermo-electro-elasto-dynamic equation for piezoelectric hollow structures is decomposed into a quasi-static homogeneous equation with inhomogeneous boundary conditions and an inhomogeneous dynamic equation with homogeneous boundary conditions. Using the method described by Lekhnitskii (1981), the quasi-static question can be exactly solved. The solution to the inhomogeneous dynamic question which satisfies homogeneous boundary conditions is obtained by utilizing the corresponding finite Hankel transforms (Cinelli, 1965), and the Laplace transforms. Then, using an interpolation method, Volterra integral equation (Kress, 1989) of the second kind caused by interaction between thermo-elastic field and thermo-electric field is solved. Thus, the exact expressions for the transient responses of displacement, stresses, electric displacement and electric potential in piezoelectric hollow structures are obtained.

In Section 2, based on spherical coordinates, the governing equation of thermo-electro-elastic transient responses in a transversely isotropic piezoelectric hollow sphere is found and the associated numerical results are carried out. In Section 3, based on cylindrical coordinates, the governing equation of thermo-electro-elastic transient responses in a non-homogeneous piezoelectric hollow cylinder is found and the corresponding numerical results are carried out. The results carried out may be used as a reference to solve other transient coupled problems of thermo-electro-elasticity in piezoelectric hollow structures.

2. Thermo-electro-elastic transient responses in a transversely isotropic piezoelectric hollow sphere

2.1. The constitutive relation and governing equation

A spherical coordinate system (r, θ, φ) with the origin identical to the center of a hollow sphere is used. For the spherically symmetric problem, we have $u_\theta = u_\varphi = 0$, $u_r = u_r(r, t)$. For a transversely isotropic piezoelectric hollow sphere subjected to a rapid change in temperature $T(r, t)$, the thermo-electro-elastic transient responses of the hollow sphere is a spherically symmetric problem, so that he constitutive relations of a spherically transversely isotropic pyroelectric medium are expressed as (Sinha, 1962; Chen and Shioya, 2001)

$$\sigma_{rr} = c_{11}\frac{\partial u_r}{\partial r} + 2c_{12}\frac{u_r}{r} + e_{11}\frac{\partial \psi}{\partial r} - \lambda_{11}T(r, t) \tag{2.1a}$$

$$\sigma_{\theta\theta} = c_{12}\frac{\partial u_r}{\partial r} + (c_{22} + c_{23})\frac{u_r}{r} + e_{12}\frac{\partial \psi}{\partial r} - \lambda_{12}T(r, t) \tag{2.1b}$$

$$D_{rr} = e_{11}\frac{\partial u_r}{\partial r} + 2e_{12}\frac{u_r}{r} - \beta_{11}\frac{\partial \psi}{\partial r} + p_{11}T(r, t) \tag{2.1c}$$

$$\lambda_{11} = c_{11}\alpha_r + 2c_{12}\alpha_\theta, \quad \lambda_{12} = c_{12}\alpha_r + (c_{22} + c_{23})\alpha_\theta \tag{2.1d}$$

where c_{ij}, e_{ij}, α_i, β_{ij}, and p_{11} are elastic constants, piezoelectric constants, thermal expansion coefficients, dielectric constants, and pyroelectric coefficients, respectively. σ_{ii} and D_{rr} are the component of stress and radial electric displacement, respectively. The equation of motion is expressed as

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{2(\sigma_{rr} - \sigma_{\theta\theta})}{r} = \rho\frac{\partial^2 u_r(r,t)}{\partial t^2} \tag{2.2}$$

where ρ is the mass density. In absence of free charge density, the charge equation of electro-statics is

$$\frac{\partial D_{rr}(r,t)}{\partial r} + \frac{2D_{rr}(r,t)}{r} = 0 \tag{2.3}$$

In order to simplify calculation, the following non-dimensional forms are introduced:

$$c_i = \frac{c_{i2}}{c_{11}} \ (i=1,2), \quad c_3 = \frac{c_{23}}{c_{11}}, \quad e_i = \frac{e_{1i}}{\sqrt{c_{11}\beta_{11}}} \ (i=1,2), \quad \lambda_i = \frac{\lambda_{1i}}{\alpha_r c_{11}} \ (i=1,2)$$

$$p_1 = \frac{p_{11}}{\alpha_r\sqrt{c_{11}\beta_{11}}}, \quad \sigma_i = \frac{\sigma_{ii}}{\alpha_r T_0 c_{11}} \ (i=r,\theta), \quad \phi = \sqrt{\frac{\beta_{11}}{c_{11}}}\frac{\psi}{b\alpha_r T_0}, \quad D_r = \frac{D_{rr}}{\alpha_r T_0\sqrt{c_{11}\beta_{11}}} \tag{2.4}$$

$$T_1(\xi,\tau) = \frac{T(r,t)}{T_0}, \quad u = \frac{u_r}{\alpha_r T_0 b}, \quad \xi = \frac{r}{b}, \quad s = \frac{a}{b}, \quad C_V = \sqrt{\frac{c_{11}}{\rho}}, \quad \tau = \frac{C_V t}{b}$$

Then, Eqs. (2.1–2.3) can be rewritten as

$$\sigma_r = \frac{\partial u}{\partial\xi} + 2c_1\frac{u}{\xi} + e_1\frac{\partial\phi}{\partial\xi} - \lambda_1 T_1(\xi,\tau) \tag{2.5a}$$

$$\sigma_\theta = c_1\frac{\partial u}{\partial\xi} + (c_2 + c_3)\frac{u}{\xi} + e_2\frac{\partial\phi}{\partial\xi} - \lambda_2 T_1(\xi,\tau) \tag{2.5b}$$

$$D_r = e_1\frac{\partial u}{\partial\xi} + 2e_2\frac{u}{\xi} - \frac{\partial\phi}{\partial\xi} + p_1 T_1(\xi,\tau) \tag{2.5c}$$

$$\frac{\partial\sigma_r}{\partial\xi} + \frac{2(\sigma_r - \sigma_\theta)}{\xi} = \frac{\partial^2 u(\xi,\tau)}{\partial\tau^2} \tag{2.6a}$$

$$\frac{\partial D_r(\xi,\tau)}{\partial\xi} + \frac{2D_r}{\xi} = 0 \tag{2.6b}$$

where, a and b are the inner and outer radii of the hollow sphere, respectively, and T_0 is the reference temperature, the boundary conditions and the initial conditions are

$$\sigma_r(s,\tau) = N_s(\tau) \quad \sigma_r(1,\tau) = N_1(\tau) \tag{2.6c,d}$$

$$\phi(s,\tau) = \phi_s(\tau), \quad \phi(1,\tau) = \phi_1(\tau) \tag{2.6e,f}$$

$$[u(\xi,\tau)]_{\tau=0} = 0, \quad \left[\frac{\partial u(\xi,\tau)}{\partial\tau}\right]_{\tau=0} = 0 \tag{2.6g,h}$$

From Eq. (2.6b), we have

$$D_r(\xi,\tau)=\frac{1}{\xi^2}d(\tau) \tag{2.7}$$

where $d(\tau)$ is an undetermined function to non-dimensional time τ.

Substituting Eq. (2.7) into Eq. (2.5c), gives

$$\frac{\partial\phi}{\partial\xi}=e_1\frac{\partial u}{\partial\xi}+2e_2\frac{u}{\xi}-\frac{1}{\xi^2}d(\tau)+p_1T_1(\xi,\tau) \tag{2.8}$$

Utilizing Eq. (2.8), Eqs. (2.5a) and (2.5b) may rewritten as

$$\sigma_r=(1+e_1^2)\frac{\partial u}{\partial\xi}+2(c_1+e_1e_2)\frac{u}{\xi}-\frac{e_1}{\xi^2}d(\tau)-(\lambda_1-e_1p_1)T_1(\xi,\tau) \tag{2.9a}$$

$$\sigma_\theta=(c_1+e_1e_2)\frac{\partial u}{\partial\xi}+(c_2+c_3+2e_2^2)\frac{u}{\xi}-\frac{e_2}{\xi^2}d(\tau)-(\lambda_2-e_2p_1)T_1(\xi,\tau) \tag{2.9b}$$

Substituting Eqs. (2.9) into Eq. (2.6a), the basic displacement equation of thermo-electro-elastic motion of a transversely isotropic piezoelectric hollow sphere is expressed as

$$\frac{\partial^2u(\xi,\tau)}{\partial\xi^2}+\frac{2}{\xi}\frac{\partial u(\xi,\tau)}{\partial\xi}-\frac{H^2u(\xi,\tau)}{\xi^2}=\frac{1}{C_L^2}\frac{\partial^2u(\xi,\tau)}{\partial\tau^2}+I\frac{d(\tau)}{\xi^3}+g(\xi,\tau) \tag{2.10}$$

where

$$H^2=\frac{2(c_2+c_3+2e_2^2-c_1-e_1e_2)}{m},\quad C_L^2=m,\quad I=-\frac{2e_2}{m},\quad m=1+e_1^2,$$

$$g(\xi,\tau)=\frac{1}{m}\left[(\lambda_1-e_1p_1)\frac{\partial T}{\partial\xi}+2(\lambda_1-\lambda_2+(e_2-e_1)p_1)\frac{T}{\xi}\right] \tag{2.11}$$

where the detailed solution process for Eq. (2.10) is given in Appendix A.

2.2. Numerical results of the first example and discussions

Transient responses and distributions of a transversely isotropic piezoelectric hollow sphere subjected to thermal shock, sudden mechanical load and electric excitation are, respectively, considered. A transitory temperature change produced by a sudden electric current pulse is typically of a duration much less than 1 μs and may be expressed as

$$T_1(\xi,\tau)=H(\tau) \tag{2.12}$$

where $H(\tau)$ denotes the Heaviside function.

In the numerical calculations, the internal radius of the transversely isotropic piezoelectric hollow sphere is taken as $a=0.01$ m, the material constants for the transversely isotropic piezoelectric hollow sphere are selected as follows:

$$c_{11}=c_{33}=111.0\text{ GPa}\quad c_{22}=125.6\text{ GPa},\quad c_{12}=77.8\text{ GPa},\quad c_{13}=c_{23}=74.3\text{ GPa},$$
$$e_{11}=15.1\ (\text{C/m}^2),\quad e_{12}=e_{13}=-5.2\ (\text{C/m}^2),\quad \alpha_r=2.0\times10^{-5}\ (1/\text{k}),$$
$$\alpha_\theta=2.0\times10^{-6}(1/\text{k}),\quad p_{11}=-2.5\times10^{-5}\ (\text{C/m}^2\,\text{k}),\quad \beta_{11}=5.62\times10^{-9}\ (\text{C}^2/\text{N}\,\text{m}^2),$$

In the example, the ratio to wall thickness is $s = 1/2$. The dimensionless time is taken as $\tau 1 = \frac{C_L \tau}{(1-s)C_v} = \frac{C_L t}{b-a}$, the dimensionless radial coordinate, $R1 = \frac{\xi - s}{1-s} = \frac{r-a}{b-a}$ and the response time is taken as $\tau 1 = 10$. Besides the thermal shock load, $T(\xi, \tau)$, in Eq. (2.12), a sudden mechanical load and electric excitation load are given by

$$\begin{aligned} &\sigma_r(s,\tau) = H(\tau) \quad \sigma_r(1,\tau) = 0 \\ &\phi_s(s,\tau) = 0 \quad \phi_1(1,\tau) = H(\tau) \end{aligned} \tag{2.13}$$

From Figs. 1, 2 and 5, it is seen that the radial stresses and the electric potential at the boundaries $R = 0$, 1 satisfy the given boundary conditions. Because of the reflection wave effect between the internal and external boundaries, except the points at given boundary condition, transient responses at other points oscillates dramatically as shown in Figs. 1–5. It is seen from Figs. 1 and 2 that the radial stress at $R1 = 0.1$ near the internal boundary appears in equal amplitude oscillatory around zero and the radial stress at $R1 = 0.5$ oscillates below zero. The maximum amplitude of radial compression stress is smaller than that of radial tension stress. It is peculiar that because of thermo-electro-elastic interactions the hoop stress at the

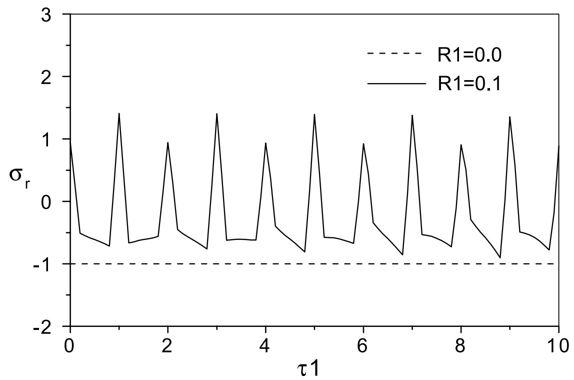

Fig. 1. Response histories of the transient radial stresses in a transversely isotropic piezoelectric hollow sphere, where $R1 = \frac{r-a}{b-a}$, $\tau 1 = \frac{C_L t}{b-a}$.

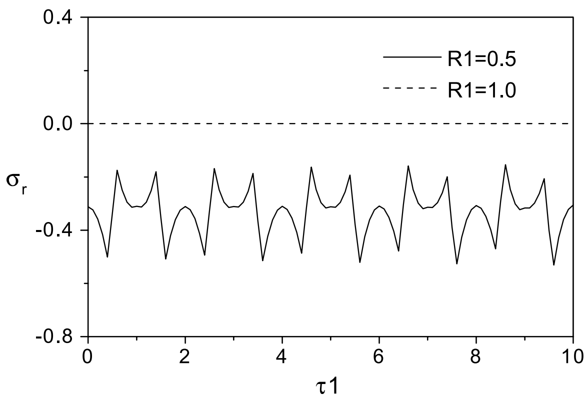

Fig. 2. Response histories of the transient radial stresses in a transversely isotropic piezoelectric hollow sphere, where $R1 = \frac{r-a}{b-a}$, $\tau 1 = \frac{C_L t}{b-a}$.

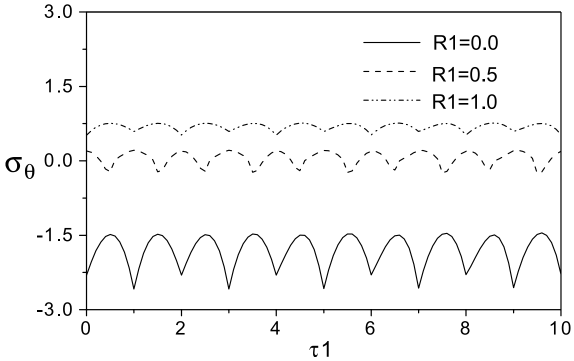

Fig. 3. Response histories of the transient hoop stresses in a transversely isotropic piezoelectric hollow sphere, where $R1 = \frac{r-a}{b-a}$, $\tau 1 = \frac{C_L t}{b-a}$.

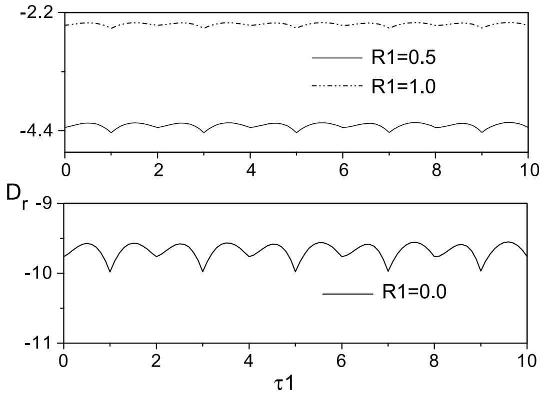

Fig. 4. Response histories of the transient electric displacements in a transversely isotropic piezoelectric hollow sphere, where $R1 = \frac{r-a}{b-a}$, $\tau = \frac{C_L t}{b-a}$.

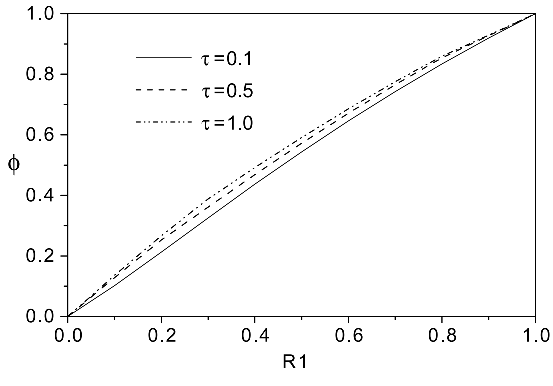

Fig. 5. Distributions of the transient electric potentials in a transversely isotropic piezoelectric hollow sphere, where $R1 = \frac{r-a}{b-a}$, $\tau = \frac{C_L t}{b-a}$.

external boundary of the transversely isotropic piezoelectric hollow sphere under sudden external pressure appears tensional stress response which is different from response history of hoop stress in a transversely isotropic hollow sphere under only mechanical load. Fig. 3 shows that the oscillatory amplitude of hoop compression stress in the transversely isotropic piezoelectric hollow sphere is smaller than that of hoop tension stress. The response histories of electric displacement always are negative as shown in Fig. 4. It is seen from Fig. 4 that the peak values of electric displacement decrease gradually from inner-wall to outer wall at the identical time $\tau 1$. The distribution of the electric potential in the transversely isotropic piezoelectric hollow sphere is shown in Fig. 5. It is seen in Fig. 5 that the electric potential at the external boundary equals 1, which satisfy the prescribed electric boundary conditions (2.13), and the distribution of the electric potential along radius is weak non-linear as time $\tau 1$.

3. Thermo-electro-elastic transient response in a non-homogeneous orthotropic piezoelectric hollow cylinder

3.1. The constitutive relation and governing equation

A cylinder coordinate system (r, θ, z) is used for an axisymmetric problem, so that we have $u_\theta = u_\varphi = 0$, $u_r = u_r(r, t)$. The constitutive relation of a long non-homogeneous orthotropic piezoelectric hollow cylinder subjected to a rapid change in temperature, $T(r, t)$, and electric fields is expressed as

$$\sigma_{rr} = c_{11}\frac{\partial u_r}{\partial r} + c_{12}\frac{u_r}{r} + e_{11}\frac{\partial \psi(r,t)}{\partial r} - \lambda_1 T(r,t) \tag{3.1a}$$

$$\sigma_{\theta\theta} = c_{12}\frac{\partial u_r}{\partial r} + c_{22}\frac{u_r}{r} + e_{12}\frac{\partial \psi(r,t)}{\partial r} - \lambda_2 T(r,t) \tag{3.1b}$$

$$D_{rr} = e_{11}\frac{\partial u_r}{\partial r} + e_{12}\frac{u_r}{r} - \beta_{11}\frac{\partial \psi(r,t)}{\partial r} + p_{11} T(r,t) \tag{3.1c}$$

$$\lambda_1 = c_{11}\alpha_1 + c_{12}\alpha_2 + c_{13}\alpha_3, \quad \lambda_2 = c_{12}\alpha_1 + c_{22}\alpha_2 + c_{23}\alpha_3, \quad \lambda_3 = c_{13}\alpha_1 + c_{23}\alpha_2 + c_{33}\alpha_3 \tag{3.1d}$$

In the above formula, the non-homogeneity of material is characterized by a special law as follows:

$$\begin{aligned} &c_{ij} = R^{2N} C_{ij}\ (i,j = 1,2,3), \quad e_{1i} = R^{2N} E_{1i}\ (i = 1,2,3), \quad \lambda_i = R^{2N} A_i\ (i = 1,2,3), \\ &\beta_{11} = R^{2N} B_{11}, \quad p_{11} = R^{2N} P_{11}, \quad \rho = R^{2N}\rho_0, \quad R = r/b \end{aligned} \tag{3.2}$$

where C_{ij}, E_{1i}, A_i, B_{11}, P_{11} and ρ_0 are known constants of homogeneous material, and N can be an arbitrary real number. $\sigma_{ii}(i = r, \theta)$ and D_{rr} are the component of stress and radial electric displacement, respectively. The equation of motion is expressed as

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \rho\frac{\partial^2 u_r(r,t)}{\partial t^2} \tag{3.3}$$

In absence of free charge density, the charge equation of electro-statics is

$$\frac{\partial D_{rr}(r,t)}{\partial r} + \frac{D_{rr}}{r} = 0 \tag{3.4}$$

Substituting Eq. (3.2) into Eqs. (3.1), (3.3) and (3.4), gives

$$\sigma_r = R^{2N}\left[\frac{\partial u}{\partial R} + C_1\frac{u}{R} + E_1\frac{\partial \phi}{\partial R} - T_1(R,\tau)\right] \tag{3.5a}$$

$$\sigma_\theta = R^{2N}\left[C_1\frac{\partial u}{\partial R} + C_2\frac{u}{R} + E_2\frac{\partial\phi}{\partial R} - T_2(R,\tau)\right] \tag{3.5b}$$

$$D_r = R^{2N}\left[E_1\frac{\partial u}{\partial R} + E_2\frac{u}{R} - \frac{\partial\phi}{\partial R} + T_p(R,\tau)\right] \tag{3.5c}$$

$$\frac{\partial\sigma_r}{\partial R} + \frac{\sigma_r - \sigma_\theta}{R} = R^{2N}\frac{\partial^2 u}{\partial\tau^2} \tag{3.5d}$$

$$\frac{\partial D_r(R,\tau)}{\partial R} + \frac{D_r}{R} = 0 \tag{3.5e}$$

where

$$C_1 = \frac{C_{12}}{C_{11}},\quad C_2 = \frac{C_{22}}{C_{11}},\quad C_3 = \frac{C_{13}}{C_{11}},\quad C_4 = \frac{C_{23}}{C_{11}},\quad E_i = \frac{E_{1i}}{\sqrt{C_{11}B_{11}}}\ (i=1,2),$$

$$\sigma_i = \frac{\sigma_{ii}}{C_{11}}\ (i=r,\theta),\quad \phi = \sqrt{\frac{B_{11}}{C_{11}}}\frac{\psi(r,t)}{b},\quad D_r = \frac{D_{rr}}{\sqrt{C_{11}B_{11}}},\quad T_i(R,\tau) = \frac{A_iT(r,t)}{C_{11}}\ (i=1,2) \tag{3.6}$$

$$T_p(R,\tau) = \frac{P_{11}T(r,t)}{\sqrt{C_{11}B_{11}}},\quad u = \frac{u_r}{b}\quad S = \frac{a}{b},\quad C_V = \sqrt{\frac{C_{11}}{\rho_0}},\quad \tau = \frac{C_Vt}{b},$$

The boundary conditions are expressed as

$$\sigma_r(S,\tau) = 0\quad \sigma_r(1,\tau) = 0 \tag{3.7a,b}$$

$$\phi(S,\tau) = \phi_a(\tau)\quad \phi(1,\tau) = \phi_b(\tau) \tag{3.7c,d}$$

where $\phi_a(\tau)$ and $\phi_b(\tau)$ are the given electric potential imposed on the internal and external surfaces, respectively.

The initial conditions are given by

$$u(R,0) = 0,\quad \dot{u}(R,\tau) = 0\quad \text{at } \tau = 0, \tag{3.8a,b}$$

where a dot over a quantity denotes its partial derivative with respect to time.

Solving Eq. (3.5e), yields

$$D_r(R,\tau) = \frac{1}{R}d(\tau) \tag{3.9}$$

where $d(\tau)$ is an undetermined function with respect to the dimensionless time τ.

Substituting Eq. (3.9) into Eq. (3.5), yields

$$\frac{\partial\phi}{\partial R} = E_1\frac{\partial u}{\partial R} + E_2\frac{u}{R} - \frac{1}{R^{2N+1}}d(\tau) + T_p(R,\tau) \tag{3.10a}$$

$$\sigma_r = R^{2N}\left[(1+E_1^2)\frac{\partial u}{\partial R} + (C_1 + E_1E_2)\frac{u}{R} - T_{1p}(R,\tau)\right] - \frac{E_1}{R}d(\tau) \tag{3.10b}$$

$$\sigma_\theta = R^{2N}\left[(C_1 + E_1E_2)\frac{\partial u}{\partial R} + (C_2 + E_2^2)\frac{u}{R} - T_{2p}(R,\tau)\right] - \frac{E_2}{R}d(\tau) \tag{3.10c}$$

where

$$\begin{aligned} T_{1p}(R,\tau) &= T_1(R,\tau) - E_1 T_p(R,\tau), \quad T_{2p}(R,\tau) = T_2(R,\tau) - E_2 T_p(R,\tau) \\ T_{3p}(R,\tau) &= T_3(R,\tau) - E_3 T_p(R,\tau) \end{aligned} \tag{3.10d}$$

Substituting Eqs. (3.10b) and (3.10c) into Eq. (3.5d), the basic displacement equation of thermo-electro-elastic motion of a non-homogeneous orthotropic piezoelectric hollow cylinder subjected to thermal shock is expressed as

$$\frac{\partial^2 u(R,\tau)}{\partial R^2} + (2N+1)\frac{1}{R}\frac{\partial u(R,\tau)}{\partial R} - \frac{H_1^2 u(R,\tau)}{R^2} = \frac{1}{C_L^2}\frac{\partial^2 u(R,\tau)}{\partial \tau^2} + I\frac{d(\tau)}{R^{2(N+1)}} + G_1(R,\tau) \tag{3.11}$$

where

$$\begin{aligned} H_1 &= \sqrt{\frac{(C_2+E_2^2) - 2N(C_1+E_1E_2)}{1+E_1^2}}, \quad C_L = \sqrt{1+E_1^2}, \quad I = -\frac{E_2}{1+E_1^2}, \\ G_1(R,\tau) &= \frac{1}{1+E_1^2}\left[\frac{\partial T_{1p}}{\partial R} + \frac{1}{R}(T_{1p-2p})\right] \end{aligned} \tag{3.12}$$

where the detailed solution process for Eq. (3.11) is given in Appendix B.

3.2. Numerical results of the second example and discussions

Transient responses and distributions of stresses, electric displacement and electric potential in a non-homogeneous orthotropic piezoelectric hollow cylinder subjected to thermal shock and a suddenly electric potential on the external surface are, respectively, considered. A transitory temperature change produced by a sudden electric current pulse is a typically of duration much less than 1 μs, and can be expressed as

$$T(r,t) = H(t) \tag{3.13}$$

where $H(t)$ denotes the Heaviside function.

In the numerical calculations, the basic material constants are taken as: $C_{11} = C_{33} = 111.0$ [GPa], $C_{22} = 220.0$ [GPa],$C_{12} = 77.8$ [GPa], $C_{13} = C_{23} = 115.0$ [GPa], $E_{12} = 15.1$ [C/m^2], $E_{12} = 15.1$ [C/m^2], $E_{11} = E_{13} = -5.2$ [C/m^2], $\alpha_1 = \alpha_3 = 0.0001$ [1/K], $\alpha_2 = 0.00001$ [1/K], $\beta_{11} = 5.62 \times 10^{-9}$ [C^2/N m^2], $P_{11} = -2.5 \times 10^{-5}$ [N/m^2 K] and $\rho_0 = 4350$ [kg/m^3]. In results, the internal radii of the non-homogeneous piezoelectric hollow cylinder are taken as $a = 0.01$ m, and the wall thickness ratio is taken as $S = 1/2$. The dimensionless time is taken as $\tau 1 = \frac{C_L \tau}{(1-S)C_v} = \frac{C_L t}{b-a}$, the dimensionless radial coordinate, $R1 = \frac{R-S}{1-S} = \frac{r-a}{b-a}$ and the response time is taken as $\tau 1 \leqslant 20$.

The non-homogeneous piezoelectric hollow cylinder is subjected to thermal shock load, $T(r,t)$, in Eq. (3.13), and an electric excitation which is expressed as

$$\phi(s,\tau) = 0, \quad \phi_b(1,\tau) = H(\tau) \tag{3.14}$$

Fig. 6 shows the response histories of the radial stress at the middle point ($R1 = 0.5$) of the piezoelectric hollow cylinder for different N. From the curves, it is seen that the maximum amplitudes of radial stresses in the piezoelectric hollow cylinder subjected to thermal shock and an electric excitation vary as different material exponent N. Figs. 7 and 8 show the responses of the hoop stresses at the internal and external surfaces, respectively. From the curves, it is seen that the peak values of the hoop stresses decreases as the radial point $R1$ increases. Figs. 9 and 10 show the radial electric displacements at the internal boundary ($R1 = 0.0$) and the external boundary ($R1 = 1.0$) of the hollow cylinder for different values of N. It is seen that the response histories of the electric displacement in the piezoelectric hollow cylinder always appear in

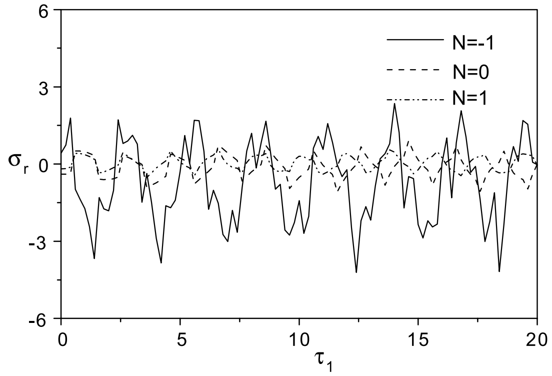

Fig. 6. Transient response histories of the radial stresses in a non-homogeneous orthotropic piezoelectric hollow cylinder for different N, where $R_1 = 0.5$.

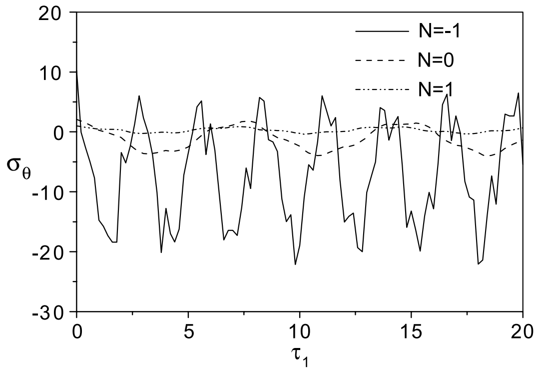

Fig. 7. Transient response histories of the hoop stresses in a non-homogeneous orthotropic piezoelectric hollow cylinder for different N, where $R_1 = 0$.

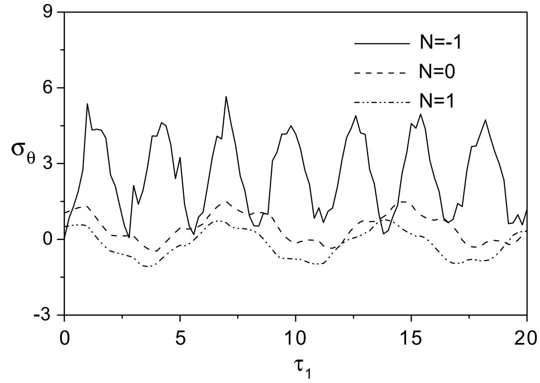

Fig. 8. Transient response histories of the hoop stresses in a non-homogeneous orthotropic piezoelectric hollow cylinder for different N, where $R_1 = 1$.

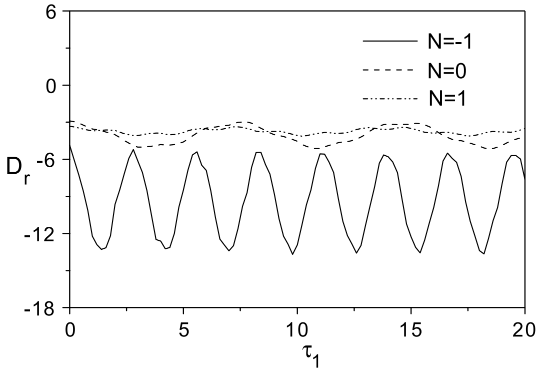

Fig. 9. Transient response histories of the electric displacements in a non-homogeneous orthotropic piezoelectric hollow cylinder for different N, where $R_1 = 0$.

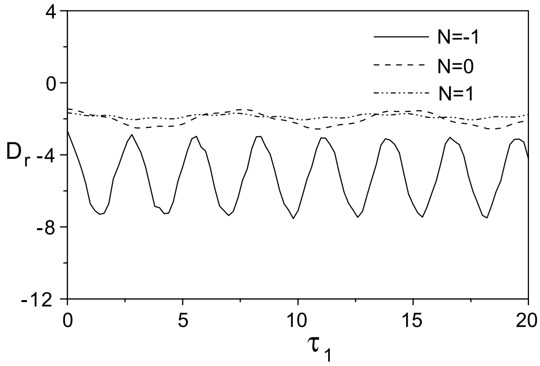

Fig. 10. Transient response histories of the electric displacements in a non-homogeneous orthotropic piezoelectric hollow cylinder for different N, where $R_1 = 1.0$.

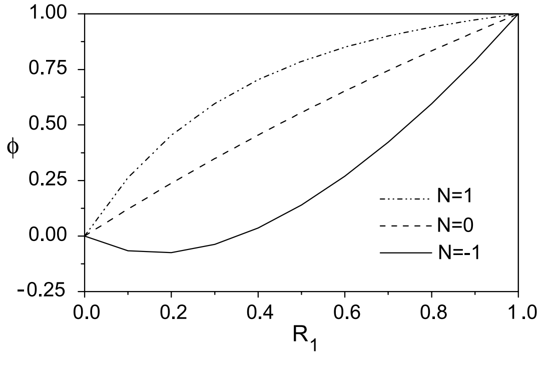

Fig. 11. Distributions of the transient electric potentials in a non-homogeneous orthotropic piezoelectric hollow cylinder for different N, where $\tau 1 = 10$.

negative. Fig. 11 illustrates the distributions of electric potential at the response time, $\tau 1 = 10$, for different material exponent N. The calculated electric potentials equal zero both at the internal and external surfaces, which satisfies with the prescribed electric boundary conditions.

4. Conclusions

1. In example 1, it is seen that the response histories and distributions of stresses, electric displacement and electric potential in a transversely isotropic piezoelectric hollow sphere are interaction each other. Thus, it is possible to control the response histories and distribution of thermal stresses in the transversely isotropic piezoelectric hollow sphere by applying a suitable mechanical load or electric excitation load to the structure, or to assessment the response histories and distribution of thermal stresses in the transversely isotropic piezoelectric hollow sphere by measuring the response histories of electric potential in the structure.
2. In example 2, the non-homogeneity of material is characterized by N value based on the basic material constants. For $N = 0$ the material property of the orthotropic piezoelectric hollow cylinder is homogeneity. For $N \neq 0$ the material property of the orthotropic piezoelectric hollow cylinder is non-homogeneity. The thermo-electro-elastic responses in the orthotropic piezoelectric hollow cylinder are mainly dependent on non-homogeneity properties of material. Therefore, one can design the non-homogeneity property, N, of piezoelectric structures to decrease the amplitude of stresses and to increase the response amplitude of electric signal in the piezoelectric structures in order to satisfy the requirement of engineering applications.
3. Because of the interaction between elastic deformation and electric field, a sudden mechanical load induces the response of electric displacement and electric potential in a piezoelectric structure. Likewise, a sudden electric potential also causes the dynamic stresses responses. Thus, applying a suitable electric excitation to a piezoelectric structure can control the responses and distributions of dynamic stresses in the piezoelectric structure.
4. Though it may be convenient and straightforward that to employ a numerical solution (Finite element method) solves some problems, it is more effective that to employ an analytical method exactly describe the interaction effect of thermo-electro-elastic waves and the effect of the non-homogeneity properties on thermo-electro-elastic transient response in piezoelectric structures subjected to arbitrary thermal shock, sudden mechanical load and electric excitation.
5. It is concluded from the above analyses and results that the present method is simple and validated. So it can be used as a reference to solve other transient problems of the coupled thermo-electro-elasticity. From the knowledge of the response histories of transient stresses, electric displacement and electric potential in piezoelectric structures, various thermo-electro-elastic elements under thermal shock load, sudden mechanical load and transient electric excitation can be designed to meet specific engineering requirements.

Acknowledgments

The authors wish to thank the supports of National Science Foundations of China (19972041, 10272075) and the referees for their valuable comments.

Appendix A

The general solution to the governing equation (2.10) of thermo-electro-elastic motion in a transversely isotropic piezoelectric hollow sphere can be decomposed into

$$u(\xi,\tau)=u_q(\xi,\tau)+u_d(\xi,\tau) \tag{A.1}$$

where $u_q(\xi,\tau)$ and $u_d(\xi,\tau)$ are, respectively, a quasi-static solution which satisfies inhomogeneous boundary conditions and a dynamic solution which satisfies homogeneous boundary conditions, to Eq. (2.10).

The quasi-static solution $u_q(\xi,\tau)$ must satisfy the following equation (A.2a) and the corresponding inhomogeneous boundary conditions (A.2b).

$$\frac{\partial^2 u_q(\xi,\tau)}{\partial\xi^2}+\frac{2}{\xi}\frac{\partial u_q(\xi,\tau)}{\partial\xi}-\frac{H^2}{\xi^2}u_q(\xi,\tau)=I\frac{d(\tau)}{\xi^3}+g(\xi,\tau) \tag{A.2a}$$

$$\left[\frac{\partial u_q(\xi,\tau)}{\partial\xi}+h\frac{u_q(\xi,\tau)}{\partial\xi}\right]_{\xi=j}=\theta_j(\tau)\quad (j=s,1) \tag{A.2b}$$

Solving Eqs. (A.2) (Lekhnitskii, 1981), we have

$$u_q(\xi,\tau)=A_1(\xi,\tau)+A_2(\xi)N_s(\tau)+A_3(\xi)N_1(\tau)+A_4(\xi)d(\tau) \tag{A.3}$$

where

$$\begin{aligned}
&A_1(\xi,\tau)=g_1(\xi,\tau)+L_1L_3\xi^{n-0.5}+L_2L_4\xi^{-(n+0.5)}\\
&A_2(\xi)=\frac{L_1}{1+e_1^2}\xi^{n-0.5}+\frac{L_2}{1+e_1^2}\xi^{-(n+0.5)}\\
&A_3(\xi)=\frac{L_1}{1+e_1^2}s^{-n-1.5}\xi^{n-0.5}+\frac{L_2}{1+e_1^2}s^{n-1.5}\xi^{-(n+0.5)}\\
&A_4(\xi)=L_1L_5\left[\frac{1}{s^2}-s^{-(n+1.5)}\right]\xi^{n-0.5}+L_2L_5\left[\frac{1}{s^2}-s^{(n-0.5)}\right]\xi^{-(n+0.5)}-\frac{I}{H^2\xi}\\
&n=\sqrt{0.25+H^2},\quad g_1(\xi,\tau)=\xi^{-n-0.5}\int_s^{\xi}\xi^{2n-1}\left[\int_s^{\xi}\xi^{-n+1.5}g(\xi,\tau)\mathrm{d}\xi\right]\mathrm{d}\xi\\
&g_2(\xi,\tau)=g_1'(\xi,\tau)+\frac{h}{\xi}g(\xi,\tau),\quad L_1=\frac{1}{(n+h-0.5)[s^{n-0.5}-s^{-(n+1.5)}]}\\
&L_2=\frac{1}{(n-h+0.5)[s^{n-0.5}-s^{-(n+1.5)}]},\quad L_5=\frac{e_1}{1+e_1^2}+\frac{(h-1)I}{H^2}\\
&L_3=\frac{1}{m}[T_{1p}(s,\tau)-T_{1p}(1,\tau)s^{-(n+1.5)}]-[g_2(s,\tau)-g_2(1,\tau)s^{-(n+1.5)}]\\
&L_4=\frac{1}{m}[T_{1p}(s,\tau)-T_{1p}(1,\tau)s^{n-0\cdots5}]-[g_2(s,\tau)-g_2(1,\tau)s^{n-0\cdots5}]
\end{aligned} \tag{A.4}$$

Substituting Eq. (A.1) into Eq. (2.10) and Eqs. (2.6g,h) and utilizing Eq. (A.2) provides an inhomogeneous dynamic equation with homogeneous boundary conditions, and the corresponding initial conditions for $u_d(\xi,\tau)$

$$\frac{\partial^2 u_d(\xi,\tau)}{\partial\xi^2}+\frac{2}{\xi}\frac{\partial u_d(\xi,\tau)}{\partial\xi}-\frac{H^2}{\xi^2}u_d(\xi,\tau)=\frac{1}{C_L^2}\left[\frac{\partial^2 u_d(\xi,\tau)}{\partial\tau^2}+\frac{\partial^2 u_q(\xi,\tau)}{\partial\tau^2}\right] \tag{A.5a}$$

$$\left[\frac{\partial u_d(\xi,\tau)}{\partial\xi}+h\frac{u_d(\xi,\tau)}{\xi}\right]_{\xi=j}=0\quad (j=s,1) \tag{A.5b,c}$$

$$u_d(\xi,0)+u_q(\xi,0)=0,\quad \frac{\partial u_d(\xi,0)}{\partial\tau}+\frac{\partial u_q(\xi,0)}{\partial\tau}=0 \tag{A.5d}$$

In order to transform Eq. (A.5a) into a normal Bessel equation, a new dependent variable $f(\xi,\tau)$ is introduced as

$$u_d(\xi,\tau)=\xi^{-0.5}f(\xi,\tau) \tag{A.6}$$

Then Eq. (A.5) is rewritten as

$$\frac{\partial^2 f(\xi,\tau)}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial f(\xi,\tau)}{\partial\xi}-\frac{R^2}{\xi^2}f(\xi,\tau)=\frac{1}{C_L^2}\left[\frac{\partial^2 f(\xi,\tau)}{\partial\tau^2}+\frac{\partial^2 u_{q1}(\xi,\tau)}{\partial\tau^2}\right] \tag{A.7a}$$

$$\frac{\partial f(s,\tau)}{\partial\xi}+h_s f(s,\tau)=0,\quad \frac{\partial f(1,\tau)}{\partial\xi}+h_1 f(1,\tau)=0 \tag{A.7b,c}$$

$$f(\xi,0)=0,\quad \frac{\partial f(\xi,0)}{\partial\tau}=0 \tag{A.7d,e}$$

where

$$u_{q1}(\xi,\tau)=B_1(\xi,\tau)+B_2(\xi)N_s(\tau)+B_3(\xi)N_1(\tau)+B_4(\xi)d(\tau),\quad h_i=\frac{(h-0.5)}{i}\ (i=s,1)$$

$$R^2=0.25+H^2,\quad B_1(\xi,\tau)=\xi^{0.5}A_1(\xi,\tau),\quad B_i(\xi)=\xi^{0.5}A_i(\xi),(i=2,3,4) \tag{A.8}$$

Define a finite Hankle transform $f(r,t)$ such that (Cinelli, 1965)

$$\bar{f}(k_i,\tau)=H[f(\xi,\tau)]=\int_s^1\xi f(\xi,\tau)G_R(k_i\xi)\mathrm{d}\xi \tag{A.9}$$

Then the inverse Hankle transform is given by

$$f(\xi,\tau)=\sum_{k_i}\frac{\bar{f}(k_i,\tau)}{F(k_i)}G_R(k_i\xi) \tag{A.10}$$

where

$$F(k_i)=\int_s^1\xi[G_R(k_i\xi)]^2\mathrm{d}\xi,\quad G_R(k_i\xi)=J_R(k_i\xi)Y_s-J_sY_R(k_i\xi) \tag{A.11}$$

In the above formula k_i $(i=1,2,\ldots,n)$ are a series of positive roots of the natural eigenequation as follows:

$$J_sY_1-J_1Y_s=0 \tag{A.12}$$

and

$$\begin{aligned}J_s&=k_iJ'_R(k_is)+h_sJ_R(k_is),\quad J_1=k_iJ'_R(k_i)+h_1J_R(k_i)\\Y_s&=k_iY'_R(k_is)+h_sY_R(k_is),\quad Y_1=k_iY'_R(k_i)+h_1Y_R(k_i)\end{aligned} \tag{A.13}$$

where $J_R(k_i\xi)$ and $Y_R(k_i\xi)$ are the first and the second kind of the Rth-order Bessel function, respectively. $\omega_i=C_L\,\mathrm{k}_i$ expresses the natural frequencies.

Applying the finite Hankel transform (A.9) to Eq. (A.7a) and utilizing the corresponding boundary condition (A.7b,c), we have

$$-k_i^2\bar{f}(k_i,\tau)=\frac{1}{C_L^2}\left[\frac{\partial^2\bar{f}(k_i,\tau)}{\partial\tau^2}+\frac{\partial^2\bar{u}_{q1}(k_i,\tau)}{\partial\tau^2}\right] \tag{A.14a}$$

where

$$\bar{u}_{q1}(k_i,\tau)=H[u_{q1}(\xi,\tau)] \tag{A.14b}$$

Applying the Laplace transform to the two sides of Eq. (A.14a) and utilizing the zero initial conditions (A.7d,e), yields

$$-k_i^2 C_L^2 \bar{f}^*(k_i,p) = p^2 \bar{f}^*(k_i,p) + p^2 \bar{u}_{q1}^*(k_i,p) \tag{A.15}$$

where p is the parameter of the Laplace transform.

The inverse Laplace transform for Eq. (A.15) gives

$$\bar{f}(k_i,\tau) = -\bar{u}_{q1}(k_i,\tau) + \omega_i[\bar{u}_{q1}(k_i,\tau)\sin(\omega_i\tau)] \tag{A.16}$$

where

$$\bar{u}_{q1}(k_i,\tau)\sin(\omega_i\tau) = \int_0^\tau [\overline{B}_1(k_i,t) + \overline{B}_2(k_i)N_s(t) + \overline{B}_3(k_i)N_1(t) + \overline{B}_4(k_i)d(t)]\sin[\omega_i(\tau-t)]\mathrm{d}t$$

and

$$\overline{B}_1(k_i,\tau) = H[B_1(\xi,\tau)], \quad \overline{B}_j(k_i) = H[B_j(\xi)] \quad (j=2,3,4) \tag{A.17}$$

Substituting Eq. (A.17) into Eq. (A.16), yields

$$\bar{f}(k_i,\tau) = I_{1i}(k_i,\tau) + \sum_{j=2}^{4}\overline{B}_j(k_i)I_{ji}(k_i,\tau) \tag{A.18}$$

where

$$\begin{aligned} I_{1i}(k_i,\tau) &= -\overline{B}_1(k_i,\tau) + \omega_i\int_0^\tau \overline{B}_1(k_i,t)\sin[\omega_i(\tau-t)]\mathrm{d}t \\ I_{2i}(k_i,\tau) &= -N_s(\tau) + \omega_i\int_0^\tau N_s(t)\sin[\omega_i(\tau-t)]\mathrm{d}t \\ I_{3i}(k_i,\tau) &= -N_1(\tau) + \omega_i\int_0^t N_1(t)\sin[\omega_i(\tau-t)]\mathrm{d}t \\ I_{4i}(k_i,\tau) &= -d(\tau) + \omega_i\int_0^\tau d(t)\sin[\omega_i(\tau-t)]\mathrm{d}t \end{aligned} \tag{A.19}$$

Substituting Eq. (A.18) into Eq. (A.10), the dynamic solution for inhomogeneous dynamic equation (A.7a) with homogeneous boundary conditions can be obtained as follows:

$$f(\xi,\tau) = \sum_{k_i}\frac{G_R(k_i\xi)}{F(k_i)}\left[I_{1i}(k_i,\tau) + \sum_{j=2}^{4}\overline{B}_j(k_i)I_{ji}(k_i,\tau)\right] \tag{A.20}$$

From Eqs. (A.20), (A.6), (A.3) and (A.1), the solution of the basic displacement equation (2.10) of thermo-electro-elastic motion in the piezoelectric hollow sphere is expressed as

$$\begin{aligned} u(\xi,\tau) &= A_1(\xi,\tau) + A_2(\xi)N_s(\tau) + A_3(\xi)N_1(\tau) + A_4(\xi)d(\tau) \\ &\quad + \sum_{k_i}\frac{\xi^{-0.5}G_R(k_i\xi)}{F(k_i)}\left[I_{1i}(k_i,\tau) + \sum_{j=2}^{4}\overline{B}_j(k_i)I_{ji}(k_i,\tau)\right] \end{aligned} \tag{A.21}$$

Noting that in the above expression $d(\tau)$ still is an unknown function which is related to the electric displacement. Thus, it is necessary to determine $\mathrm{d}(\tau)$ in the following.

Integrating Eq. (2.5c) and utilizing the corresponding electric boundary condition (2.6e), yields

$$\phi(\xi,\tau) = \Phi_1(\xi,\tau) + \Phi_2(\xi)N_s(\tau) + \Phi_3(\xi)N_1(\tau) + \Phi_4(\xi)d(\tau) + \sum_i \Phi_{5i}(\xi)F_i(\tau) + \phi_s(\tau) \tag{A.22}$$

where

$$\Phi_1(\xi,\tau) = e_1\left[A_1(\xi,\tau) - A_1(s,\tau) - \sum_{k_i}\frac{(\xi^{-0.5}G_R(k_i\xi) - s^{-0.5}G_R(k_is))}{F(k_i)}\overline{B}_1(k_i,\tau)\right] + 2e_2\int_s^{\xi}\frac{1}{\xi}\left[A_1(\xi,\tau) - \sum_{k_i}\frac{\xi^{-0.5}G_R(k_i\xi)}{F(k_i)}\overline{B}_1(k_i,\tau)\right]\mathrm{d}\xi + p_1\int_s^{\xi}T(\xi,\tau)\mathrm{d}\xi \tag{A.23a}$$

$$\Phi_2(\xi) = e_1\left[A_2(\xi) - A_2(s) - \sum_{k_i}\frac{(\xi^{-0.5}G_R(k_i\xi) - s^{-0.5}G_R(k_is))}{F(k_i)}\overline{B}_2(k_i)\right] + 2e_2\int_s^{\xi}\frac{1}{\xi}\left[A_2(\xi) - \sum_{k_i}\frac{\xi^{-0.5}G_R(k_i\xi)}{F(k_i)}\overline{B}_2(k_i)\right]\mathrm{d}\xi \tag{A.23b}$$

$$\Phi_3(\xi) = e_1\left[A_3(\xi) - A_3(s) - \sum_{k_i}\frac{(\xi^{-0.5}G_R(k_i\xi) - s^{-0.5}G_R(k_is))}{F(k_i)}\overline{B}_3(k_i)\right] + 2e_2\int_s^{\xi}\frac{1}{\xi}\left[A_3(\xi) - \sum_{k_i}\frac{\xi^{-0.5}G_R(k_i\xi)}{F(k_i)}\overline{B}_3(k_i)\right]\mathrm{d}\xi \tag{A.23c}$$

$$\Phi_4(\xi) = e_1\left[A_4(\xi) - A_4(s) - \sum_{k_i}\frac{(\xi^{-0.5}G_R(k_i\xi) - s^{-0.5}G_R(k_is))}{F(k_i)}\overline{B}_4(k_i)\right] + 2e_2\int_s^{\xi}\frac{1}{\xi}\left[A_4(\xi) - \sum_{k_i}\frac{\xi^{-0.5}G_R(k_i\xi)}{F(k_i)}\overline{B}_4(k_i)\right]\mathrm{d}\xi + \frac{1}{\xi} \tag{A.23d}$$

$$\Phi_{5i}(\xi) = e_1\frac{(\xi^{-0.5}G_R(k_i\xi) - s^{-0.5}G_R(k_is))}{F(k_i)} + 2e_2\int_s^{\xi}\frac{1}{\xi^{1.5}}\frac{G_R(k_i\xi)}{F(k_i)}\mathrm{d}\xi \tag{A.23e}$$

$$F_i(\tau) = F_{1i}(\tau) + \overline{B}_4(k_i)\omega_i\int_0^{\tau}d(t)\sin[\omega_i(\tau - t)]\mathrm{d}t \tag{A.23f}$$

$$F_{1i}(\tau) = \omega_i\int_0^{\tau}\overline{B}_1(k_i,t)\sin[\omega_i(\tau - t)]\mathrm{d}t + \overline{B}_2(k_i)\omega_i\int_0^{\tau}N_s(t)\sin[\omega_i(\tau - t)]\mathrm{d}t + \overline{B}_3(k_i)\omega_i\int_0^{\tau}N_1(t)\sin[\omega_i(\tau - t)]\mathrm{d}t \tag{A.23g}$$

When $\xi = 1$ at the outer boundary of the transversely isotropic piezoelectric hollow sphere, Eq. (A.22) is rewritten as

$$\phi_1(\tau) = \Phi_1(1,\tau) + \Phi_2(1)N_s(\tau) + \Phi_3(1)N_1(\tau) + \Phi_4(1)d(\tau) + \sum_i\Phi_{5i}(1)F_i(\tau) + \phi_s(\tau) \tag{A.24}$$

Substituting $\tau = 0$ into Eq. (A.24), leads to

$$d(0) = \frac{\phi_1(0) - \phi_s(0) - \Phi_1(1,0) - \Phi_2(1)N_s(0) - \Phi_3(1)N_1(0) - \sum_i\Phi_{5i}(1)F_i(0)}{\Phi_4(1)} \tag{A.25}$$

Substituting Eq. (A.23f) into Eq. (A.24), yields

$$\vartheta(\tau) = M_1 d(\tau) + \sum_i M_{2i} \int_0^\tau d(t)\sin[\omega_i(\tau - t)]\mathrm{d}t \tag{A.26}$$

where

$$\begin{aligned} \vartheta(\tau) &= \phi_1(\tau) - \phi_s(\tau) - \Phi_1(1,\tau) - \Phi_2(1)N_s(\tau) - \Phi_3(1)N_1(\tau) - \sum_i \Phi_{5i}(1)F_{1i}(\tau) \\ M_1 &= \Phi_4(1), \quad M_{2i} = \Phi_{5i}(1)\overline{B}_4(k_i)\omega_i \end{aligned} \tag{A.27}$$

It is seen that Eq. (A.26) is the Volterra integral equation (Kress, 1989) of the second kind. In the following, we will solve Eq. (A.26) by using the recursion formula based on linear interpolation function. In order to show the method of solving the integral equation (A.26), the time interval $[0,\tau]$ is divided into n subintervals, that is, the discrete time points are $\tau_0 = 0, \tau_1, \tau_2, \ldots, \tau_n$. Then the interpolation function at the time interval $[\tau_{j-1}, \tau_j]$ is expressed as

$$d(\tau) = \eta_j^0(\tau)d(\tau_{j-1}) + \eta_j^1(\tau)d(\tau_j) \quad (j = 1, 2, \ldots, n) \tag{A.28a}$$

where

$$\eta_j^0(\tau) = \frac{\tau - \tau_j}{\tau_{j-1} - \tau_j}, \quad \eta_j^1(\tau) = \frac{\tau - \tau_{j-1}}{\tau_j - \tau_{j-1}} \quad (j = 1, 2, \ldots, n) \tag{A.28b}$$

Substituting Eq. (A.28) into Eq. (A.26), gives

$$\vartheta(\tau_j) = M_1 d(\tau_j) + \sum_i M_{2i} \sum_{k=1}^{j} [R_{ijk} d(\tau_{k-1}) + S_{ijk} d(\tau_k)] \tag{A.29}$$

where

$$R_{ijk} = \int_{\tau_{k-1}}^{\tau_k} \eta_k^0(t)\sin[\omega_i(\tau - t)]\mathrm{d}t \tag{A.30a}$$

$$S_{ijk} = \int_{\tau_k - 1}^{\tau_k} \eta_k^1(t)\sin[\omega_i(\tau - t)]\mathrm{d}t \quad (k = 1, 2, \ldots, j, \ j = 1, 2, \ldots, n) \tag{A.30b}$$

From Eq. (A.29), we have

$$d(\tau_j) = \frac{\vartheta(\tau_j) - \sum_i M_{2i} \sum_{k=1}^{j-1} [R_{ijk} d(\tau_{k-1}) + S_{ijk} d(\tau_k)] - d(\tau_{j-1}) \sum_i M_{2i} R_{ijj}}{M_1 + \sum_i M_{2i} S_{ijj}} \quad (j = 1, 2, \ldots, n) \tag{A.31}$$

Substituting $d(0)$ in Eq. (A.25) into Eq. (A.26), we can obtain $d(\tau_j)$ $(j = 1, 2, \ldots, n)$ step by step, and determine $d(\tau)$. Substituting $d(\tau)$ obtained from Eq. (A.31) into Eq. (A.21), gives the exact expression of the solution, $u(\xi,\tau)$, for the basic equation of thermo-electro-elastic equation (2.10) in the transversely isotropic piezoelectric hollow sphere. Thus, the corresponding transient stresses $\sigma_r(\xi,\tau)$, $\sigma_\theta(\xi,\tau)$, the transient electric displacement $D_r(\xi,\tau)$ and the transient electric potential $\phi(\xi,\tau)$ are easily obtained from Eqs. (2.5a–c), (A.22), and (A.21).

Appendix B

A new dependent variable $W(R,\tau)$ is introduced as

$$u(R,\tau) = R^{-N} W(R,\tau) \tag{B.1}$$

Utilizing the Eqs. (B.1), Eqs. (3.11), (3.7) and (3.8) are rewritten as

$$\frac{\partial^2 W}{\partial R^2} + \frac{1}{R}\frac{\partial W}{\partial R} - \frac{H^2 W}{R^2} = \frac{1}{C_L^2}\frac{\partial^2 W}{\partial \tau^2} + I\frac{d(\tau)}{R^{N+2}} + G_2(R,\tau) \tag{B.2a}$$

$$R = S: \quad \frac{\partial W(R,\tau)}{\partial R} + h\frac{W(R,\tau)}{R} = \theta_1(\tau) \tag{B.2b}$$

$$R = 1: \quad \frac{\partial W(R,\tau)}{\partial R} + h\frac{W(R,\tau)}{R} = \theta_2(\tau) \tag{B.2c}$$

$$W(R,0) = 0 \quad \dot{W}(R,0) = 0 \tag{B.2d,e}$$

where

$$H = \sqrt{N^2 + H_1^2}, \quad h = \frac{C_1 + E_1 E_2}{1 + E_1^2} - N, \quad G_2(R,\tau) = R^N G_1(R,\tau)$$

$$\theta_1(\tau) = \frac{S^N}{1 + E_1^2}\left[E_1\frac{d(\tau)}{S^{(2N+1)}} + T_{1p}(S,\tau)\right], \quad \theta_2(\tau) = \frac{1}{1 + E_1^2}[E_1 d(\tau) + T_{1p}(1,\tau)]$$

$$-u_1(R) = R^N u_0(R), \quad v_1(R) = R^N v_0(R). \tag{B.3a–g}$$

The general solution to the governing equation (B.2) can be decomposed into

$$W(R,\tau) = W_q(R,\tau) + W_d(R,\tau) \tag{B.4}$$

Here, the quasi-static solution $W_q(R,\tau)$ must satisfy the following equations:

$$\frac{\partial^2 W_q(R,\tau)}{\partial R^2} + \frac{1}{R}\frac{\partial W_q(R,\tau)}{\partial R} - \frac{H^2 W_q(R,\tau)}{R^2} = I\frac{d(\tau)}{R^{N+2}} + G_2(R,\tau) \tag{B.5a}$$

$$R = S: \quad \frac{\partial W_q(R,\tau)}{\partial R} + h\frac{W_q(R,\tau)}{R} = \theta_1(\tau) \tag{B.5b}$$

$$R = 1: \quad \frac{\partial W_q(R,\tau)}{\partial R} + h\frac{W_q(R,\tau)}{R} = \theta_2(\tau) \tag{B.5c}$$

The general integral of Eqs. (B.5a) is of the form

$$W_q(R,\tau) = \psi_1(R,\tau) + \psi_2(R)d(\tau) \tag{B.6}$$

In the above formula, we have

$$\psi_1(R,\tau) = R^{-H}\int_S^R R^{2H-1}\int_S^R R^{-H+N+1}G_2(R,\tau)\mathrm{d}R\,\mathrm{d}R + \left[\frac{R^{2H} - S^{2H}}{2H^2} - \frac{S^{2H}}{h - H}\right]R^{-H}q_3(\tau) + \frac{S^{N+H+1}T_{1p}(S,\tau)}{(h - H)C_L^2}R^{-H} \tag{B.7a}$$

$$\psi_2(R) = IR^{-H}\int_S^R R^{2H-1}\int_S^R R^{-(H+N+1)}\,\mathrm{d}R\,\mathrm{d}R + g_3\left[\frac{R^{2H}-S^{2H}}{2H^2} - \frac{S^{2H}}{h-H}\right]R^{-H} + \frac{E_1 S^{H-N}}{(h-H)C_L^2}R^{-H} \tag{B.7b}$$

$$g_3 = g_2\left[(H-h)g_1 + \frac{I(1-S^{-(H+N)})}{H+N} + \frac{E_1}{C_L^2}(1-S^H)\right],$$

$$g_2 = \frac{2H}{(1-S^{2H})(H+h)},\quad g_1 = -\frac{I}{N+H}\left(\frac{1-S^{N-H}}{N-H} + \frac{1-S^{2H}}{2H}\right),$$

$$q_3(\tau) = g_4\left\{(H-h)q_1(\tau) - q_2(\tau) + \frac{1}{C_L^2}[T_{1p}(1,\tau) - S^{H+1}T_{1p}(S,\tau)]\right\},$$

$$q_1(\tau) = \int_S^1 R^{2H-1}\int_S^R R^{-H+N+1}G_2(R,\tau)\,\mathrm{d}R\,\mathrm{d}R,\quad q_2(\tau) = \int_S^1 R^{-H+N+1}G_2(R,\tau)\mathrm{d}R \tag{B.8c–h}$$

Substituting (B.4) into (B.2) and utilizing (B.5) provides an inhomogeneous dynamic equation with homogeneous boundary conditions for $W_d(R,\tau)$

$$\frac{\partial^2 W_d(R,\tau)}{\partial R^2} + \frac{1}{R}\frac{\partial W_d(R,\tau)}{\partial R} - \frac{H^2}{R^2}W_d(R,\tau) = \frac{1}{C_L^2}\left[\frac{\partial^2 W_d(R,\tau)}{\partial\tau^2} + \frac{\partial^2 W_q(R,\tau)}{\partial\tau^2}\right] \tag{B.9a}$$

$$\left[\frac{\partial W_d(R,\tau)}{\partial R} + h\frac{W_d(R,\tau)}{R}\right]_{R=S} = 0,\quad \left[\frac{\partial W_d(R,\tau)}{\partial R} + h\frac{W_d(R,\tau)}{R}\right]_{R=1} = 0 \tag{B.9b}$$

$$W_d(R,0) + W_q(R,0) = u_1(R)\quad \frac{\partial W_d(R,0)}{\partial\tau} + \frac{\partial W_q(R,0)}{\partial\tau} = v_1(R) \tag{B.9c,d}$$

In the above equation, $W_q(R,\tau)$ is the known solution as shown in Eq. (B.6).

Utilizing the solving processes of Eqs. (A.9)–(A.31) as shown in Appendix A, the solution, $W_d(R,\tau)$, for Eq. (B.9) can be easily obtained. Thus, the exact expression of the solution $u(R,\tau)$ is given for the governing equation of thermo-electro-elastic motion in a non-homogeneous orthotropic piezoelectric hollow cylinder. The corresponding transient stresses $\sigma_r(R,\tau)\sigma_\theta\,(R,\tau)$, the transient electric displacement $D_r(R,\tau)$ and the transient electric potential $\phi(R,\tau)$ are easily obtained.

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